Thermodynamic formalism for some systems with countable Markov structures

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To the memory of Dmitry Viktorovich Anosov

Abstract

We study ergodic properties of certain piecewise smooth two-dimensional systems by constructing countable Markov partitions. Using thermodynamic formalism we prove exponential decay of correlations for Hölder functions. That extends previous results of M. Jakobson and S. Newhouse (2000), where Bernoulli property was proved for such systems. Our approach is motivated by the original method of D.V. Anosov and Ya.G. Sinai (1967).

1 Motivation: Folklore Theorem in dimension 1

A well-known Folklore Theorem in one-dimensional dynamics can be formulated as follows.

Folklore Theorem. *Let* $I = [0,1]$ *be the unit interval, and suppose* $\{I_1, I_2, \dots\}$ *is a countable collection of disjoint open subintervals of* I *such that* $\bigcup_i I_i$ *has the full Lebesgue measure in* I*. Suppose there are constants* $K_0 > 1$ *and* $K_1 > 0$ *and mappings* $f_i : I_i \to I$ *satisfying the following conditions.*

1. f_i *extends to a* C^2 *diffeomorphism from the closure of* I_i *onto* $[0,1]$*, and* $\inf_{z \in I_i} | Df_i(z) | > K_0$ *for all* i*.*

2. $\sup_{z \in I_i} \frac{| D^2 f_i(z) |}{| Df_i(z) |} | I_i | < K_1$ *for all* i*.*

Then, the mapping $F(z)$ defined by $F(z) = f_i(z)$ for $z \in I_i$, has a unique invariant ergodic probability measure μ equivalent to Lebesgue measure on I.

For the proof of the Folklore theorem , the ergodic properties of μ and the history of the question see for example [4] and [18].
In [9] , [10] the Folklore Theorem was generalized to two-dimensional maps F which piecewise coincide with certain hyperbolic diffeomorphisms f_i. As in the one-dimensional situation there is an essential difference between a finite and an infinite number of f_i. In the case of an infinite number of f_i, their derivatives grow with i and relations between first and second derivatives become crucial.
Models with infinitely many f_i appear when we study non-hyperbolic systems, such as quadratic-like maps in dimension 1, and Henon-like maps in dimension 2.

2 Model under consideration. Geometric and hyperbolicity conditions

1. As in [9] , [10] we consider the following 2-d model. Let Q be the unit square. Let $\xi = \{E_1, E_2, \dots, \}$ be a countable collection of closed curvilinear rectangles in Q. Assume that each E_i lies inside a domain of definition of a C^2 diffeomorphism f_i which maps E_i onto its image $S_i \subset Q$. We assume each E_i connects the top and the bottom of Q. Thus each E_i is bounded from above and from below by two subintervals of the line segments $\{(x,y) : y = 1,\ 0 \le x \le 1\}$ and $\{(x,y) : y = 0,\ 0 \le x \le 1\}$. Hyperbolicity conditions that we formulate below imply that the left and right boundaries of E_i are graphs of smooth functions $x^{(i)}(y)$ with $\left|\frac{dx^{(i)}}{dy}\right| \le \alpha$ where α is a real number satisfying $0 < \alpha < 1$.
   The images $f_i(E_i) = S_i$ are narrow strips connecting the left and right sides of Q and that they are bounded on the left and right by the two subintervals of the line segments $\{(x,y) : x = 0,\ 0 \le y \le 1\}$ and $\{(x,y) : x = 1,\ 0 \le y \le 1\}$ and above and below by the graphs of smooth functions $Y^i(X), \mid \frac{dY^{(i)}}{dX} \mid \le \alpha$.
   We are saying that $E_i^{\prime}s$ are *full height* in Q while the $S_i^{\prime}s$ are *full width* in Q.

2. For $z \in Q$, let ℓ_z be the horizontal line through z. We define $\delta_z(E_i) = diam(\ell_z \bigcap E_i)$, $\delta_{i,max} = \max_{z \in Q} \delta_z(E_i)$, $\delta_{i,min} = \min_{z \in Q} \delta_z(E_i)$. We assume the following

Geometric conditions.

G1. For $i \neq j$ holds $int\ E_i \cap int\ E_j = \emptyset$ and $int\ S_i \cap int\ S_j = \emptyset$.

G2. $mes(Q \setminus \cup_i\ int\, E_i) = 0$ where *mes* stands for Lebesgue measure.

G3. $-\sum_i \delta_{i,max} \log \delta_{i,min} < \infty$.

3. In the standard coordinate system for a map $F : (x,y) \to (F_1(x,y), F_2(x,y))$ we use $DF(x,y)$ to denote the differential of F at some point (x,y) and F_{jx}, F_{jy}, F_{jxx}, F_{jxy}, etc., for partial derivatives of F_j, $j = 1,2$.

   Let $J_F(z) = \mid F_{1x}(z)F_{2y}(z) - F_{1y}(z)F_{2x}(z) \mid$ be the absolute value of the Jacobian determinant of F at z.

Hyperbolicity conditions.

There exist constants $0 < \alpha < 1$ and $K_0 > 1$ such that for each i the map

$$F(z) = f_i(z) \text{ for } z \in E_i$$

satisfies

H1. $\mid F_{2x}(z) \mid + \alpha \mid F_{2y}(z) \mid + \alpha^2 \mid F_{1y}(z) \mid \leq \alpha \mid F_{1x}(z) \mid$

H2. $\mid F_{1x}(z) \mid - \alpha \mid F_{1y}(z) \mid \geq K_0$.

H3. $\mid F_{1y}(z) \mid + \alpha \mid F_{2y}(z) \mid + \alpha^2 \mid F_{2x}(z) \mid \leq \alpha \mid F_{1x}(z) \mid$

H4. $\mid F_{1x}(z) \mid - \alpha \mid F_{2x}(z) \mid \geq J_F(z) K_0$.

For a real number $0 < \alpha < 1$, we define the cones

$$K^u_\alpha = \{(v_1, v_2) : \mid v_2 \mid \leq \alpha \mid v_1 \mid\}$$

$$K^s_\alpha = \{(v_1, v_2) : \mid v_1 \mid \leq \alpha \mid v_2 \mid\}$$

and the corresponding cone fields $K^u_\alpha(z), K^s_\alpha(z)$ in the tangent spaces at points $z \in \mathbf{R}^2$.

The following proposition proved in [10] relates conditions H1-H4 above with the usual definition of hyperbolicity in terms of cone conditions. It shows that conditions H1 and H2 imply that the K^u_α cone is mapped into itself by DF and expanded by a factor no smaller than K_0 while H3 and H4 imply that the K^s_α cone is mapped into itself by DF^{-1} and expanded by a factor no smaller than K_0.

Unless otherwise stated, we use the *max* norm on $\mathbf{R}^2$, $|(v_1,v_2)| = \max(|v_1|,|v_2|)$.

Proposition 2.1 *Under conditions H1-H4 above, we have*

$$DF(K^u_\alpha) \subseteq K^u_\alpha \tag{1}$$

$$v \in K^u_\alpha \Rightarrow |DFv| \geq K_0|v| \tag{2}$$

$$DF^{-1}(K^s_\alpha) \subseteq K^s_\alpha \tag{3}$$

$$v \in K^s_\alpha \Rightarrow |DF^{-1}v| \geq K_0|v| \tag{4}$$

Remark 2.2 The first version of hyperbolicity conditions appeared in [17]. It was developed in particular in [5] and [8] . Here we use hyperbolicity conditions from [10]. In [9] we used hyperolicity conditions from [5] which implied the invariance of cones and uniform expansion with respect to the sum norm $|v| = |v_1| + |v_2|$.

4. The map

$$F(z) = f_i(z) \text{ for } z \in int\ E_i$$

is defined almost everywhere on Q. Let $\tilde{Q}_0 = \bigcup_i int\ E_i$, and, define $\tilde{Q}_n, n > 0$, inductively by $\tilde{Q}_n = \tilde{Q}_0 \bigcap F^{-1}\tilde{Q}_{n-1}$. Let $\tilde{Q} = \bigcap_{n\geq 0}\tilde{Q}_n$ be the set of points whose forward orbits always stay in $\bigcup_i int\ E_i$. Then, $\tilde{Q}$ has full Lebesgue measure in Q, and F maps $\tilde{Q}$ into itself.

The hyperbolicity conditions H1–H4 imply the estimates on the derivatives of the boundary curves of E_i and S_i which we described earlier. They also imply that any intersection $f_iE_i \bigcap E_j$ is full width in E_j. Further, $E_{ij} =$

$E_i \bigcap f_i^{-1} E_j$ is a full height subrectangle of E_i and $S_{ij} = f_j f_i E_{ij}$ is a full width substrip in Q.

Given a finite string $i_0 \dots i_{n-1}$, we define inductively

$$E_{i_0 \dots i_{n-1}} = E_{i_0} \bigcap f_{i_0}^{-1} E_{i_1 i_2 \dots i_{n-1}}.$$

Then, each set $E_{i_0 \dots i_{n-1}}$ is a full height subrectangle of E_{i_0}.

Analogously, for a string $i_{-m} \dots i_{-1}$ we define

$$S_{i_{-m} \dots i_{-1}} = f_{i_{-1}} (S_{i_{-m} \dots i_{-2}} \bigcap E_{i_{-1}})$$

and get that $S_{i_{-m} \dots i_{-1}}$ is a full width strip in Q. It is easy to see that $S_{i_{-m} \dots i_{-1}} = f_{i_{-1}} \circ f_{i_{-2}} \circ \dots \circ f_{i_{-m}} (E_{i_{-m} \dots i_{-1}})$ and that $f_{i_0}^{-1} (S_{i_{-m} \dots i_{-1}})$ is a full-width substrip of E_{i_0}.

We also define curvilinear rectangles $R_{i_{-m} \dots i_{-1}, i_0 \dots i_{n-1}}$ by

$$R_{i_{-m} \dots i_{-1}, i_0 \dots i_{n-1}} = S_{i_{-m} \dots i_{-1}} \bigcap E_{i_0 \dots i_{n-1}}$$

If there are no negative indices then respective rectangle is full height in Q. For infinite strings, we have the following Proposition.

Proposition 2.3 *Any C^1 map F satisfying the above geometric conditions G1–G3 and hyperbolicity conditions H1–H4 has a "topological attractor" in the sense of [10]*

$$\Lambda = \bigcup_{\dots i_{-n} \dots i_{-1}} \bigcap_{k \geq 1} S_{i_{-k} \dots i_{-1}}$$

The infinite intersections $\bigcap_{k=1}^{\infty} S_{i_{-k} \dots i_{-1}}$ define C^1 curves $y(x)$, $|dy/dx| \leq \alpha$ which are the unstable manifolds for the points of the attractor. The infinite intersections $\bigcap_{k=1}^{\infty} E_{i_0 \dots i_{k-1}}$ define C^1 curves $x(y)$, $|dx/dy| \leq \alpha$ which are the stable manifolds for the points of the attractor. The infinite intersections

$$\bigcap_{m=1}^{\infty} \bigcap_{n=1}^{\infty} R_{i_{-m} \dots i_{-1}, i_0 \dots i_{n-1}}$$

define points of the attractor.

Proposition 2.3 is a well known fact in hyperbolic theory. For example it follows from Theorem 1 in [5]. See also [12]. The union of the stable manifolds has full measure in Q. The trajectories of all points in this set converge to Λ. That is the reason to call Λ a topological attractor.

5. An $F-$invariant Borel probablility measure μ on Q is called a $Sinai-Ruelle-Bowen$ measure (or SRB-measure) for F if μ is ergodic and there is a set $A \subset Q$ of positive Lebesgue measure such that for $x \in A$ and any continuous real-valued function $\phi : Q \to \mathbf{R}$, we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \phi(F^k x) = \int \phi d\mu. \tag{5}$$

Existence of an SRB measure is a much stronger result, than 2.3. It allows to describe statistical properties of trajectories in a set of positive phase volume. It requires some additional assumptions.

3 Distortion conditions

As we have a countable number of domains the derivatives of f_i grow. We formulate certain assumptions on the second derivatives. We use the distance function $d((x,y),(x_1,y_1)) = \max(\mid x-x_1 \mid, \mid y-y_1 \mid)$ associated with the norm $\mid v \mid = \max(\mid v_1 \mid, \mid v_2 \mid)$ on vectors $v = (v_1, v_2)$.

As above, for a point $z \in Q$, let l_z denote the horizontal line through z, and if $E \subseteq Q$, let $\delta_z(E)$ denote the diameter of the horizontal section $l_z \bigcap E$. We call $\delta_z(E)$ the $z-width$ of E.

In given coordinate systems we write $f_i(x,y) = (f_{i1}(x,y), f_{i2}(x,y))$. We use $f_{ijx}, f_{ijy}, f_{ijxx}, f_{ijxy}$, etc. for partial derivatives of $f_{ij}, j = 1,2$.

We define

$$\mid D^2 f_i(z) \mid = \max_{j=1,2,(k,l)=(x,x),(x,y),(y,y)} \mid f_{ijkl}(z) \mid .$$

Next we formulate distortion conditions which are used to control the fluctuation of the derivatives of iterates of F along unstable manifolds, and to construct Sinai local measures.

Suppose there is a constant $C_0 > 0$ such that the following *distortion condition* holds

D1. $\sup_{z\in E_i, i\geq 1} \frac{| D^2 f_i(z) |}{| f_{i1x}(z) |} \delta_z(E_i) < C_0$.

Our conditions imply the following theorem proved in [9], [10].

Theorem 3.1 *Let F be a piecewise smooth mapping as above satisfying the geometric conditions G1–G3, the hyperbolicity conditions H1–H4 and the distortion condition D1.*

Then, F has an SRB measure μ supported on Λ whose basin has full Lebesgue measure in Q. Dynamical system (F,μ) satisfies the following properties.

1. *(F,μ) is measure-theoretically isomorphic to a Bernoulli shift.*
2. *F has finite entropy with respect to the measure μ, and the entropy formula holds*
$$h_\mu(F) = \int \log |D^u F| d\mu \tag{6}$$
where $D^u F(z)$ is the norm of the derivative of F in the unstable direction at z.
3. $$h_\mu(F) = \lim_{n\to\infty} \frac{1}{n} \log | DF^n(z) | \tag{7}$$
where the latter limit exists for Lebesgue almost all z and is independent of such z.

4 Additional hyperbolicity and distortion conditions and statement of the main theorem

When applying thermodynamic formalism to hyperbolic attractors one considers the function $\phi(z) = -\log(D^u F(z))$. Thermodynamic formalism is based on the fact that the pullback of $\phi(z)$ into a symbolic space determined by some Markov partition is a locally Hölder function.
We prove Hölder property of $\phi(z)$ assuming an extra hyperbolicity condition, and a distortion condition D2 stronger than D1.

Hyperbolicity condition H5.

H5. $\frac{1}{K_0^2} + \alpha^2 < 1$.

Distortion condition D2.

D2. $\sup_{z \in E_i, i \geq 1} \dfrac{| D^2 f_i(z) |}{| f_{i1x}(z) |} < C_0$.

Remark 4.1 *Condition D2 is too strong to be useful for systems with critical points. In dimension 1 it reads as* $| \frac{F_{ixx}}{F_{ix}} | < c$ *instead of* $| \frac{F_{ixx}}{F_{ix}^2} | < c$. *However instead of D2 one can assume additional hyperbolicity conditions, which can be vaguely formulated as "contraction of* f_i *grows faster than expansion" . That approach will be discussed in a forthcoming paper.*

Assuming additionally H5 and D2 we prove that Hölder functions have exponential decay of correlations.
Let $\mathscr{H}_\gamma$ be the space of functions on Q satisfying Hölder property with exponent γ

$$| \phi(x) - \phi(y) | \leq c | x - y |^\gamma$$

Then the following theorem holds.

Theorem 4.2 *Let F be a piecewise smooth mapping as above satisfying the geometric conditions G1–G3, the hyperbolicity conditions H1–H5 and the distortion condition D2. Then* (F, μ) *has exponential decay of correlations for* $\phi, \psi \in \mathscr{H}_\gamma$. *Namely there exist* $\eta(\gamma) < 1$ *and* $C = C(\phi, \psi)$ *such that*

$$| \int \phi(\psi \circ F^n) d\mu - \int \phi d\mu \int \psi d\mu | < C\eta^n \tag{8}$$

5 Hölder properties of $\log(D^u F(z))$

1. Although Markov partitions are partitions of the attractor, we need to check Hölder property on actual two-dimensional curvilinear rectangles $R_{i_{-m} \dots i_{-1}, i_0 \dots i_{n-1}}$. We call respective partition Markov as well. In our model Markov partition consists of initial full height rectangles E_i.
We consider rectangles $R_{i_{-m} \dots i_{-1}, i_0 \dots i_{n-1}}$ with $m \geq 0, n \geq 1$. We use notation

$m = 0$ if there are no negative coordinates, which means $R_{i_{-m}\dots i_{-1}, i_0 \dots i_{n-1}} = R_{i_0,\dots,i_{n-1}}$ is a full height rectangle.
For any function $a(x,y)$ the variation of $a(x,y)$ over a rectangle R is defined as

$$var(a(x,y))|R = \sup_{(x_1,y_1)\in R,(x_2,y_2)\in R} | a(x_1,y_1) - a(x_2,y_2) | \tag{9}$$

By definition the function $\log D^u F$ is **locally Hölder** if for $m \geq 0$, $n \geq 1$ the variation of $\log D^u F$ on $R_{i_{-m}\dots i_{-1}, i_0 \dots i_{n-1}}$ satisfies

$$var(\log D^u F)|R_{i_{-m}\dots i_{-1}, i_0 \dots i_{n-1}} < C\theta_0^{min(m,n)} \tag{10}$$

for some $C > 0$, $\theta_0 < 1$.
The assumption $n \geq 1$, means that variations are measured between points which belong to the same full height rectangle.

Proposition 5.1 $\log D^u F$ *is a locally Hölder function.*

We prove Proposition 5.1 with some θ_0 and C determined by hyperbolicity and distortion conditions.

(a) The sets $R_{i_{-m}\dots i_{-1}, i_0 \dots i_{n-1}}$ are bounded from above and below by some arcs of two unstable curves $\Gamma^u_{i_{-m}\dots i_{-1}}$, which are images of some pieces of the top and bottom of $\tilde{Q}$, and from left and right by some arcs of two stable curves $\Gamma^s_{i_0 \dots i_{n-1}}$, which are preimages of some pieces the left and right boundaries of $\tilde{Q}$.
Let $Z_1, Z_2 \in R_{i_{-m}\dots i_{-1}, i_0 \dots i_{n-1}}$ be two points on the attractor. We connect Z_1, Z_2 by two pieces of their unstable manifolds to two points Z_3, Z_4 which belong to the same stable manifold. Let
$\gamma_1 = \gamma(Z_1, Z_3) \subset W^u(Z_1)$, $\gamma_2 = \gamma(Z_2, Z_4) \subset W^u(Z_2)$, $\gamma_3 = \gamma(Z_3, Z_4) \subset W^s(Z_3)$ be respective curves all located inside $R_{i_{-m}\dots i_{-1}, i_0 \dots i_{n-1}}$.
We estimate

$$| \log D^u F(Z_1) - \log D^u F(Z_2) | \leq | \log D^u F(Z_1) - \log D^u F(Z_3) | + \\ | \log D^u F(Z_3) - \log D^u F(Z_4) | + | \log D^u F(Z_4) - \log D^u F(Z_2) |$$

(b) First we estimate $| \log D^u F(Z_1) - \log D^u F(Z_3) |$. We cover γ_1 by a chain of small rectangles with sides parallel to the standard axes $R =$

$\Delta x \times \Delta y \subset R_{i_{-m} \ldots i_{-1}, i_0 \ldots i_{n-1}}$. Because of cone conditions we can choose rectangles $R = \Delta x \times \Delta y$ satisfying $|\, \Delta y \,| \leq \alpha |\, \Delta x \,|$. Then $|\, \log D^u F(Z_1) - \log D^u F(Z_3) \,|$ is majorated by the sum of similar differences for points $z_1, z_2 \in W^u(Z_1) \cap R$. Here z_1, z_2 are points on the vertical boundaries of R. Inside R we can use the mean value theorem.

Hyperbolicity conditions imply the following properties, see [10].

i. Any unit vector in K^u_α at a point $z \in E_i$, in particular a tangent vector to $W^u(z)$, has coordinates $(1, a_z)$ with $|\, a_z \,| < \alpha$.

ii.
$$|\, D^u F(z) \,| = |\, F_{1x}(z) + a_z F_{1y}(z) \,| \tag{11}$$

iii.
$$\frac{|\, F_{1y} \,|}{|\, F_{1x} \,|} < \alpha \tag{12}$$

iv.
$$\frac{|\, F_{2x} \,|}{|\, F_{1x} \,|} < \alpha \tag{13}$$

v.
$$\frac{|\, F_{2y} \,|}{|\, F_{1x} \,|} < \frac{1}{K_0^2} + \alpha^2 \tag{14}$$

Assuming without loss of generality $F_{1x} > 0$ for all $x \in E_i$ we get that variation of $\log |\, D^u F \,|$ between two points $z_1, z_2 \in W^u(Z_1) \subset R$ equals

$$\log \left[F_{1x}(z_1)\big(1 + a_{z_1} \frac{F_{1y}}{F_{1x}}(z_1)\big)\right] - \log \left[F_{1x}(z_2)\big(1 + a_{z_2} \frac{F_{1y}}{F_{1x}}(z_2)\big)\right] \tag{15}$$

We split it into two expressions and estimate separately

$$\log F_{1x}(z_1) - \log F_{1x}(z_2) \tag{16}$$

and
$$\log \big(1 + a_{z_1} \frac{F_{1y}}{F_{1x}}(z_1)\big) - \log \big(1 + a_{z_2} \frac{F_{1y}}{F_{1x}}(z_2)\big) \tag{17}$$

We rewrite 17 as

$$\log \big(1 + \frac{a_{z_1} \frac{F_{1y}}{F_{1x}}(z_1) - a_{z_2} \frac{F_{1y}}{F_{1x}}(z_2)}{1 + a_{z_2} \frac{F_{1y}}{F_{1x}}(z_2)}\big) \tag{18}$$

As denominator of the fraction in 18 is uniformly bounded away from 0, we estimate the numerator and rewrite it as a sum of two expressions

$$| a_{z_1} | \frac{| F_{1y}(z_1)F_{1x}(z_2) - F_{1y}(z_2)F_{1x}(z_1) |}{F_{1x}(z_1)F_{1x}(z_2)} \tag{19}$$

and

$$| a_{z_1} - a_{z_2} || \frac{F_{1y}(z_2)}{F_{1x}(z_2)} | \tag{20}$$

As C^2 sizes of unstable manifolds are uniformly bounded (see [10]), 20 is bounded by $c| \Delta x |$. We rewrite 19 as

$$a_{z_1} [\frac{F_{1y}(z_1)\big(F_{1x}(z_2) - F_{1x}(z_1)\big)}{F_{1x}(z_1)F_{1x}(z_2)} + \frac{F_{1y}(z_1) - F_{1y}(z_2)}{F_{1x}(z_2)}] \tag{21}$$

As $| \frac{F_{1y}(z_1)}{F_{1x}(z_1)} | < \alpha$, both expressions are estimated similarly.
As we are moving along W^u, we get $| \Delta y | < \alpha| \Delta x |$.
We use the mean value theorem and distortion assumptions, and get estimates bounded by

$$c| \Delta x | \frac{F_{1x}(\theta)}{F_{1x}(z_2)} \tag{22}$$

Then it remains to estimate $\frac{F_{1x}(\theta)}{F_{1x}(z_2)}$ or equivalently $| \log F_{1x}(\theta) - \log F_{1x}(z_2) |$, which is the same estimate as 16.
In order to estimate 16 we use again the mean value theorem and distortion assumptions.
Then we get

$$| \log F_{1x}(\theta) - \log F_{1x}(z_2) | < c| \Delta x | \tag{23}$$

and respectively

$$\frac{F_{1x}(\theta)}{F_{1x}(z_2)} < \exp(c| \Delta x |) \tag{24}$$

We combine the previous estimates and get

$$| \log D^u F(z_1) - \log D^u F(z_3) | < c| \gamma(z_1, z_3) | \tag{25}$$

From hyperbolicity conditions we get

$$| \gamma(z_1, z_3) | < C_2 \frac{1}{K_0^n} \tag{26}$$

That implies

$$| \log D^u F(z_1) - \log D^u F(z_3) | < C_3 \frac{1}{K_0^n} \tag{27}$$

where C_3 is a uniform constant.
Similar inequality holds for $\gamma(z_2, z_4)$.

$$| \log D^u F(z_2) - \log D^u F(z_4) | < C_3 \frac{1}{K_0^n} \tag{28}$$

(c) Next we estimate the variation of $\log | D^u F(z) |$ between points Z_3 and Z_4, which belong to the same stable manifold $W^s(Z_3) = W^s(Z_4) \subset R_{i_{-m} \dots i_{-1}, i_0 \dots i_{n-1}}$. Thus we need to estimate

$$\log | F_{1x}(Z_3) + a_{Z_3} F_{1y}(Z_3) | - \log | F_{1x}(Z_4) + a_{Z_4} F_{1y}(Z_4) | \tag{29}$$

As above we split the variation 29 into 16, 19 and 20.
This time instead of moving along $W^u(Z_1)$ we are moving along $W^s(Z_3)$, which connects Z_3 and Z_4. In that case we use $| \Delta x | < \alpha | \Delta y |$, so Δy variations are added. As above estimates 16, 19 contribute less than

$$c | \gamma_3 | < C_2 \frac{1}{K_0^m} \tag{30}$$

Next we estimate the dependence of a_z from $y \in W_0^s$.
The following lemma is sufficient for our purposes.

Lemma 5.2 *There exist $c_0 > 0$, $0 < \theta_0 < 1$ such that*

$$| a_{Z_3} - a_{Z_4} | < c_0 \theta_0^m \tag{31}$$

Proof.

We assume by induction that for any rectangle $R_{i_{-m} \dots i_{-1}, i_0 \dots i_{n-1}}$, and for any points $Z_3, Z_4 \in R_{i_{-m} \dots i_{-1}, i_0 \dots i_{n-1}}$ of intersection of two unstable manifolds W_1^u, W_2^u with the same stable manifold W_0^s, the inequality 31 holds. Then we prove

$$| a_{F(Z_3)} - a_{F(Z_4)} | < c_0 \theta_0^{m+1} \tag{32}$$

DF maps a unit vector $\vec{v} = (1, a)$ into $(F_{1x} + F_{1y}a, F_{2x} + F_{2y}a)$. Then the normalized vector $DF\vec{v}$ has second coordinate

$$a' = \frac{\frac{F_{2x}}{F_{1x}} + \frac{F_{2y}}{F_{1x}}a}{1 + \frac{F_{1y}}{F_{1x}}a} \tag{33}$$

We denote $Z_3 = z$, $Z_4 = w$ and estimate

$$\frac{\frac{F_{2x}}{F_{1x}}(z) + \frac{F_{2y}}{F_{1x}}(z)a(z)}{1 + \frac{F_{1y}}{F_{1x}}(z)a(z)} - \frac{\frac{F_{2x}}{F_{1x}}(w) + \frac{F_{2y}}{F_{1x}}(w)a(w)}{1 + \frac{F_{1y}}{F_{1x}}(w)a(w)} \tag{34}$$

After cross multiplying we get denominator bounded away from 0. Therefore it is enough to estimate two terms

$$\frac{F_{2x}}{F_{1x}}(w)\big(1 + \frac{F_{1y}}{F_{1x}}(z)a(z)\big) - \frac{F_{2x}}{F_{1x}}(z)\big(1 + \frac{F_{1y}}{F_{1x}}(w)a(w)\big) \tag{35}$$

and

$$\frac{F_{2y}}{F_{1x}}(w)a(w)\big(1 + \frac{F_{1y}}{F_{1x}}(z)a(z)\big) - \frac{F_{2y}}{F_{1x}}(z)a(z)\big(1 + \frac{F_{1y}}{F_{1x}}(w)a(w)\big) \tag{36}$$

Both expressions are estimated similarly. To estimate 36 we split it into

$$\frac{F_{2y}}{F_{1x}}(w)a(w) - \frac{F_{2y}}{F_{1x}}(z)a(z) \tag{37}$$

and

$$a(z)a(w)\big(\frac{F_{2y}}{F_{1x}}(w)\frac{F_{1y}}{F_{1x}}(z) - \frac{F_{2y}}{F_{1x}}(z)\frac{F_{1y}}{F_{1x}}(w)\big) \tag{38}$$

As above we use elementary algebra and get expressions of the type

$$\frac{F_{1x}(w) - F_{1x}(z)}{F_{1x}(z)} \tag{39}$$

and

$$\frac{F_{2y}(w) - F_{2y}(z)}{F_{1x}(z)} \tag{40}$$

We split γ_3 into small intervals, and apply the mean value theorem. The ratios $\frac{F_{1x}(\theta)}{F_{1x}(z)}$ or equivalently the differences $\log F_{1x}(\theta) - \log F_{1x}(z)$

for close points θ, z on the same stable manifold are estimated (using again the mean value theorem and D2) as

$$\log F_{1x}(\theta) - \log F_{1x}(z) < C_0(1+\alpha)\Delta y \tag{41}$$

Thus for any two points z and θ on the same stable manifold

$$\log F_{1x}(\theta) - \log F_{1x}(z) < C \mid z - \theta \mid \tag{42}$$

In particular for all points z and θ on the same stable manifold the ratios $\frac{F_{1x}(\theta)}{F_{1x}(z)}$ are uniformly bounded.
Thus estimate 38 contributes

$$C \mid \gamma_3 \mid \tag{43}$$

When estimating 37 we get similar terms estimated as 43, and

$$\frac{F_{2y}}{F_{1x}}(z)\big(a(z) - a(w)\big) \tag{44}$$

After we combine all terms except 44 we get an estimate

$$M_0 C_0 \frac{1}{K_0^m} \tag{45}$$

where M_0 is a uniform constant, which depends on the number of similar terms that we added above, and C_0 is the distortion constant from condition D2. For 44 we use inductive assumption 31 and get a total estimate

$$\mid a_{F(Z_3)} - a_{F(Z_4)} \mid < M_0 C_0 \frac{1}{K_0^m} + \big(\frac{1}{K_0^2} + \alpha^2\big) c_0 \theta_0^m \tag{46}$$

As $K_0 > 1$ we can choose $\theta_0 < 1$ satisfying

$$\theta_0 > \frac{1}{K_0} \tag{47}$$

Also H5 implies that we can choose $\theta_0 < 1$ satisfying simulteneously

$$\frac{1}{K_0^2} + \alpha^2 < \theta_0 \tag{48}$$

Then if

$$c_0 > \frac{M_0 C_0}{\theta_0 - (\frac{1}{K_0^2} + \alpha^2)} \tag{49}$$

we get the left side of 46 less than $c_0 \theta_0^{m+1}$.
Q.E.D.

Remark 5.3 *A related result for classical systems was proved by A. Pinto and D. Rand in [11]. They prove that if Λ is an invariant hyperbolic set with local product structure for a $C^{1+\gamma}$ diffeomorphism with one-dimensional unstable leaves, then holonomies between unstable leaves are $C^{1+\alpha}$ for some $\alpha > 0$.*

From Lemma 5.2 and 30 we get

$$|\log D^u F(z_3) - \log D^u F(z_4)| < C_3 \theta_0^m \tag{50}$$

Combining 27, 28, 50 we conclude the proof of Proposition 5.1.

2. We combine several corollaries from Proposition 5.1 and from the arguments used in its proof.

   Corollary 5.4 *There exists c independent of i such that for any $z_1, z_2 \in E_i$ holds*

   $$\frac{| f_{i1x}(z_1) |}{| f_{i1x}(z_2) |} < c \tag{51}$$

   Here z_1, z_2 do not need to be on the attractor.
   To prove Corollary 5.4 we fix an arbitrary stable manifold $W_i^s \subset E_i$, and connect z_1 to $z_3 \in W_i^s$ by a horizontal segment σ. As full height rectangles are bounded from above and below by horizontal segments, σ lies entirely in E_i. Similarly we connect z_2 to $z_4 \in W_i^s$. Then 23 and 42 imply 51.
   Let $\delta_z(E_i)$ be the width of the horizontal crossection of E_i through $z \in E_i \cap \Lambda$. As $\delta_z(E_i)$ are mapped onto full width unstable curves we get from 51

   Corollary 5.5 *There exists c independent of i such that for any $z_1, z_2 \in E_i$ holds*

   $$\frac{\delta_{z_1}(E_i)}{\delta_{z_2}(E_i)} < c \tag{52}$$

Remark 5.6 *Property 52 demonstrates restrictions on geometry imposed by condition D2.*
Conditions of Theorem 3.1 allow widths of E_i to oscillate exponentially between a^i and b^i for some $0 < a < b < 1$. However from 52 we get that ratios are uniformly bounded.

Applying 10 to the full height rectangles E_i we get for all $z_1, z_2 \in E_i \cap \Lambda$

Corollary 5.7

$$var(\log D^u F)|E_i < C \tag{53}$$

and

Corollary 5.8

$$\frac{D^u f_i(z_1)}{D^u f_i(z_2)} < c \tag{54}$$

For any z on an unstable curve $W^u(z) \subset E_i$ which is full width in E_i let $\mid W^u(z, E_i) \mid$ be the respective length . As $\mid W^u(z, E_i) \mid$ coincide up to a uniformly bounded factor with $\frac{1}{D^u f_i(z)}$ we get from 54

Corollary 5.9 *There exists c independent of i such that for any $z_1, z_2 \in E_i \cap \Lambda$ holds*

$$\frac{\mid W^u(z_1, E_i) \mid}{\mid W^u(z_2, E_i) \mid} < c \tag{55}$$

Note that 55 also follows from 52 because at a given point $z \in E_i$ ratios : $\frac{\delta_z(E_i)}{\mid W^u(z_1, E_i) \mid}$ are uniformly bounded.
Although the next statement is not used in the proof of the main theorem , it is usefull for understanding the geometry of partitions into $E_{i_0 \ldots i_{n-1}}$. We claim that 55 is valid for rectangles of any order.

Remark 5.10 *Let $E_{i_0 \ldots i_{n-1}}$ be a full height rectangle of order n.*
Then for any two points $z_1,\ z_2 \in E_{i_0 \ldots i_{n-1}} \cap \Lambda$ holds

$$\frac{\mid W^u(z_1, E_{i_0 \ldots i_{n-1}}) \mid}{\mid W^u(z_2, E_{i_0 \ldots i_{n-1}}) \mid} < c \tag{56}$$

To prove 56 we rewrite the ratio $\frac{D^uF^n(z_2)}{D^uF^n(z_1)}$ as

$$\frac{\prod_{i=0}^{n-1} D^uF(F^iz_2)}{\prod_{i=0}^{n-1} D^uF(F^iz_1)} \tag{57}$$

and consider

$$\sum_{i=0}^{n-1} |\log D^uF(F^iz_2) - \log D^uF(F^iz_1)| \tag{58}$$

As in the proof of Proposition 5.1 we split the estimate of each term into estimates along stable and unstable manifolds in the images $F^p(E_{i_0\ldots i_{n-1}}) = R_{i_0\ldots i_{p-1},i_p\ldots i_{n-1}}$. For each term estimates from the proof of Proposition 5.1 imply respective bounds : $C\theta_0^p$ on stable manifolds, and $C\frac{1}{K_0^{n-p}}$ on unstable manifolds. Thus we get a uniform bound in 58, which implies 56 .

3. According to 2.3 points of Λ are identified with two-sided sequences

$$(\ldots i_{-m}\ldots i_{-1}, i_0i_1\ldots i_n\ldots)$$

In order to use Ruelle-Bowen approach we define a function ϕ^u corresponding to $-\log D^uF(z)$ on the space of one-sided sequences. We fix some unstable manifold W_0^u. Let $z = (x,y) \in \Lambda$, $z_0 = W^s(z) \cap W_0^u$. For $\phi(z) = -\log D^u(z)$ define

$$u(z) = \sum_{k=0}^{\infty} \phi(F^k(z)) - \phi(F^k(z_0)) \tag{59}$$

and

$$\psi(z) = \phi(z) - u(z) + u(Fz) \tag{60}$$

From 10 we get that the series 59 converge uniformly.
For $\psi(z)$ all terms with z cancel, so $\psi(z)$ as a function of z_0 depends only on nonnegative iterates of z.

Let $\Omega_+ = \{x = (i_0i_1\ldots i_n\ldots)\}$ be the space of one-sided sequences corresponding to stable manifolds.
The above function $\psi(z)$ defined on Ω_+ will be denoted $\phi^u(x)$. On the space Ω_+ we use the metric $d(x^1,x^2) = 2^{-n}$, where $n = min\{k : i_k^1 \neq i_k^2\}$.
For a function $\phi(x)$ on the symbolic space Ω_+ let

$$V_n(\phi) = \sup|\phi(x) - \phi(y)| : x_i = y_i, i = 0,\ldots,n-1$$

$\phi(x)$ is called locally Hölder if there are $C > 0, 0 < \theta < 1$ such that $\forall n \geq 1$

$$V_n(\phi) < C\theta^n \tag{61}$$

We use the same arguments as in [7] which prove that the Hölder property on the space of two-sided sequences implies the Hölder property for respective function on the space of one-sided sequences. Then Proposition 5.1 implies

Corollary 5.11 *$\phi^u(x)$ is a locally Hölder function on the symbolic space Ω_+.*

6 Some sufficient conditions for exponential decay of correlations in countable shifts

1. We refer to [14] for the following general results about shifts with countable alphabets .
   Let T be the shift transformation on the space X of admissible one-sided sequences determined by an infinite matrix A with $a_{ij} = 0, 1$. Here i, j are states of an infinite alphabet. Let $C = [i_0, \dots, i_{n-1}]$ be cylinder sets. The system (X, A, T) is called *topologically mixing* if the following holds.

   $$\forall C_1, C_2 \;\; \exists N(C_1, C_2) : \forall n > N(C_1, C_2) \;\; C_1 \cap T^{-n} C_2 \neq \emptyset \tag{62}$$

   Let (X_A, T) be a topologically mixing countable shift, and let $\phi(x)$ be a locally Hölder function.
   Set $\phi_n(x) = \sum_{k=0}^{n-1} \phi \circ T^k(x)$. Define $Z_n(\phi, a)$ by

   $$Z_n(\phi, a) = \sum_{T^n x = x, x_0 = a} e^{\phi_n(x)} \tag{63}$$

   Then the limit called *Gurevich Pressure*

   $$P(\phi) = \lim_{n \to \infty} \frac{1}{n} \log Z_n(\phi, a) \tag{64}$$

   exists and does not depend of the choice of a, see [14].

Definition Assuming (X_A,σ) is topologically mixing and ϕ is locally Hölder continuous ϕ is called *positive recurrent* if there is $\lambda > 0$ such that for any given symbol a there is a constant $M_a > 1$ and an integer N_a such that for every $n \geq N_a$ holds

$$\frac{Z_n(\phi,a)}{\lambda^n} \in [M_a^{-1}, M_a]$$

Let L_ϕ be the Ruelle operator

$$L_\phi f(x) = \sum_{Ty=x} e^{\phi(y)} f(y) \tag{65}$$

The following is a part of Theorem 4 in [14].

Theorem 6.1 *Let (X_A,T) be a topologically mixing countable shift, and let $\phi(x)$ be a locally Hölder function such that $P(\phi) < \infty$. If ϕ is positive recurrent, then $\lambda = e^{P(\phi)}$ and there exist a σ-finite measure ν and a function $h > 0$ such that $L_\phi^* \nu = \lambda \nu$, $L_\phi h = \lambda h$, $\nu(h) = 1$, and for every uniformly continuous function f such that $||fh^{-1}||_\infty < \infty$ holds*

$$\lambda^{-n} L_\phi^n f \to \nu(f)h$$

uniformly on compacts.

Positive recurrence and convergence result in Theorem 6.1 hold for matrices A satisfying the following *Big Images and Preimages* property .

BIP There is a finite set of states $i_1, i_2, \ldots i_N$ such that for every state j in the alphabet there are k,l such that $a_{i_k j} a_{j i_l} = 1$.

The following result of Sarig from [15] extends the results of [14] and works of Aaronson, Denker , Mauldin, Urbanski and Yuri, see [2], [3], [19]. Consider the space of functions $\mathscr{L}$ with bounded norm $| f |_{\mathscr{L}}$ which is the sum of $||f||_\infty$ and some fixed Hölder norm.

Theorem 6.2 *Suppose (X_A,σ) is topologically mixing, ϕ is locally Hölder continuous, $P(\phi) < \infty$ and BIP property holds. Then*

(a) ϕ is positive recurrent, and there exist λ, h, ν as in theorem 6.1.

(b) *h is bounded away from zero and infinity and* $\nu(X) < \infty$.

(c) *There exist* $K > 0$, $\theta \in (0,1)$ *such that for* $f \in \mathscr{L}$ *holds*

$$| \lambda^{-n} L_{\phi}^{n} f - h\nu(f) |_{\mathscr{L}} < K\theta^{n} | f |_{\mathscr{L}} \tag{66}$$

2. As a corollary from Theorem 6.2 we get

Proposition 6.3 *Suppose there is a Markov partition of the attractor satisfying the following properties.*

(a) *The matrix A of admissible transitions is topologically mixing and satisfies BIP property.*

(b) $\Phi(x,y) = -log| D^{u}F |$ *is Hölder on the space of admissible sequences.*

(c) *For some* $\phi(x)$ *cohomologous to* $\Phi(x,y)$ *holds* $P(\phi(x)) < \infty$.

then 66 holds.

Proposition 6.3 gives sufficient conditions for exponential decay of correlations for Hölder (in particular smooth) functions restricted to the attractor.

7 Proof of the exponential decay of correlations

We check properties (a) - (c).

1. Recall that in our model we consider the partition of the square into full height rectangles E_i.
   Our shift is Bernoulli, all rows (and columns) are the same row of 1-s, so it is topologically mixing and property (a) is satisfied.

2. Property (b) follows from 10.

3. Next we prove property (c). As in the case of attractors for Axiom A systems we prove

   Proposition 7.1 *For* $\phi(x) = \phi^{u}(x)$ *topological pressure* $P(\phi^{u}(x))$ *equals zero.*

Proof.
We fix some symbol a, respective rectangle E_a, and $W^u_{0a} = W^u_0 \cap E_a$. When evaluating $Z_n(\phi, a)$ in 63 we consider respective sum over all periodic orbits of period n starting in E_a.
Each cylinder set $E_{ai_1\dots i_{n-1}}$ contains one periodic orbit of period n. When evaluating $\phi^u(x)$ we use formula 60. We can evaluate that expression at a point z of intersection between the stable manifold of a periodic point in $E_{ai_1\dots i_{n-1}}$ and W^u_{0a}. Then each term in 63 is a product of two expressions.
The first expression equals $\frac{1}{D^uF^n(z)}$, which coincides up to a uniformly bounded factor with the length of $W^u(z, E_{ai_1\dots i_{n-1}})$.
The second expression $e^{u(z)-u(F(z)}$ is uniformly bounded away from zero and infinity.
Therefore up to a uniformly bounded factor the sum 63 equals to the length of W^u_{0a}. That implies $P(\phi) = 0$. Q.E.D.

So all properties of Proposition 6.3 are satisfied, and we get exponential decay of correlations for one-sided shift. As in [7] it implies exponential decay of correlations for two-sided shift and therefore for Hölder functions on Q. That proves Theorem 4.2.

Remark 7.2 *Under conditions of Theorem 4.2 the central limit theorem holds for Hölder functions on Q which are not cohomologous to constants .*

Remark 7.3 *Corollary 4 from [15] implies that under conditions of Theorem 4.2 there are no "phase transitions" in the sense that the function $t \to P(-t \log D^u f)$ is real analytic in a neighborhood of $t = 1$.*

Let us denote μ_S the invariant measure on Λ constructed in [9], [10] following Sinai method, and let μ_{RB} be the invariant measure on Λ constructed above following Ruelle-Bowen method, see [13], [7]. Let μ_1 be the projection of μ_S onto one-sided sequences, and let μ be the measure on one-sided sequences constructed above by Ruelle-Bowen method. In both constructions measures of cylinder sets $[i_0 i_1 \dots i_{n-1}]$ of any rank equal up to a uniform constant to the length of the crossections of $E_{i_0\dots i_{n-1}}$ by W^u_0. So μ_1 and μ are equivalent and therefore they coincide. That is a particular case of the characterization of Gibbs measures proved in [14].
As in the classical case that implies

Corollary 7.4 *Measures μ_S and μ_{RB} coincide.*

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